Forbidden bifurcations and parametric amplification in a Josephson-junction array

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(Received 30 April 1993)

We test a general theory of parametric amplification by globally coupled arrays via numerical simulations of a shunted Josephson-junction series array operated in the three-photon mode. We find good agreement in a number of particulars: optimal amplification of periodic signals occurs near the onset of symmetry-preserving bifurcations, and gain curves follow characteristic scaling laws. We also uncover an unexpected result: the resistively shunted Josephson-junction array cannot undergo the desired bifurcation; nevertheless, substantial amplification is still possible. The response of disordered arrays is also considered.

PACS number(s): 05.45.+b, 85.25.Cp, 74.40.+k

I. INTRODUCTION

Near a bifurcation the dynamics of a system can often be simplified via a center manifold reduction, sometimes to a single variable [1]. This has led to a successful general theory of the effects of near-resonant perturbations on dynamical systems near codimension-one bifurcations [2–8]. Physically, this is of interest both for understanding the effects of noise and for exploiting the bifurcation to amplify small signals. The theory extracts scaling behavior which is independent of the physical details of the system and has been successfully applied to a diverse range of experimental systems, including magnetorestrictive ribbons [9, 10], driven bouncing balls [11, 12], NMR lasers [13], semiconductor lasers [14], parametric nuclear magnons in an antiferromagnetic crystal [15], and Josephson-junction parametric amplifiers [16].

The problem of Josephson-junction parametric amplifiers is particularly important. These have potential for several applications as small signal detectors: for radioastronomy in the millimeter and submillimeter wave regime [17, 18], the study of radiation fields of Rydberg atoms in high-Q microwave cavities [19, 20], and as back-action-evading amplifiers in Weber-bar-type gravitational wave detectors [21]. Josephson-junction parametric amplifiers have also been used to generate squeezed electromagnetic radiation, of interest in studying the interaction of squeezed states with Rydberg atoms [20, 22].

Single junctions have drawbacks, however. They have low power, low impedance, and limited dynamic range [23]. In principle these shortcomings can be overcome by linking many junctions in a series array. The abovementioned general theory cannot be applied to arrays because their bifurcations are typically more complicated than those of a single-element system. Consequently, the dynamics does not necessarily reduce to just a few active variables. Recently, the theory for perturbations near bifurcations was extended to the case of *globally coupled* oscillators, which is directly relevant to Josephson-junction series arrays [24]. Since this theory is based on a linearized analysis it must break down both sufficiently close to the bifurcation point and for sufficiently large signals. In these regimes systems display inherently nonlinear effects [6, 7], which are well understood for single-oscillator amplifiers, but not for their array counterparts [16, 25].

In this paper, we test the predictions of the linearized theory numerically, by considering a Josephson-junction series array shunted by a load. We find good agreement with the general aspects of the theory: the highest gain occurs near the onset of symmetry-preserving bifurcations, and the gain curves follow the predicted scaling laws. We also discover an unexpected but potentially important result peculiar to the Josephson array system. Namely, the resistively shunted Josephson-junction array cannot undergo an in-phase symmetry-preserving bifurcation. On the face of it, this would seem to prohibit its usefulness as a parametric amplifier, but in fact we show that substantial (though not optimal) gain is still possible under the right circumstances.

In the next section, we review the basic results of the amplifier array theory as they pertain to the Josephson-junction array example. We discuss the forbidden bifurcations in Sec. III, as well as a modification of the circuit which removes the forbidden bifurcation. The results of our numerical simulations are presented in Sec. IV, followed by a discussion in Sec. V.

II. THEORETICAL PREDICTIONS

The theory for parametric amplification by globally coupled arrays considers systems whose dynamics are described by equations of the form

$$\dot{x}_k = F\left(x_k, \sum_{j=1}^{N} x_j, \mu\right) , \quad k = 1, \ldots, N ,$$

where x_k is a vector, μ is a control parameter, and each oscillator x_k is coupled to the others only via the average behavior of the array. Equations of this form come up quite naturally in the study of series and parallel array circuits [26, 27], as well as certain solid state laser arrays [28] and multimode lasers [29–31]. For example, the Josesphson-junction array depicted in Fig. 1 is governed by the dynamical equations

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$$\ddot{\varphi}_k + \gamma\dot{\varphi}_k + \sin\varphi_k + \frac{1}{R}\sum_{j=1}^{N}\dot{\varphi}_j = I_{\rm dc} + I_{\rm ac}\cos(\omega t) + a\cos(\omega_s t + \Delta) \ , \quad k = 1, 2, \ldots, N \ . \quad (1)$$

These are the lump circuit equations in dimensionless form, using the Stewart-McCumber equivalent circuit model for the junctions [32]. Here, φ_k is the quantum phase difference across the kth junction, γ is defined as $1/\sqrt{\beta_c}$, where β_c is the McCumber parameter of the junction, R is the resistance shunting the array of junctions, expressed in units of the resistance of a single Josephson junction, and $I_{\rm dc} + I_{\rm ac}\cos(\omega t)$ is the applied bias current. The external signal is represented by the additional current source $a\cos(\omega_s t + \Delta)$.

The reason the simple single-element theory does not apply here is because of the symmetry inherent in the array problem. Mathematically, this means that the usual classification scheme [1] of bifurcations breaks down, and the problem becomes more complicated (though still tractable [33]). Physically, if the oscillators are identical, or nearly identical, the clean separation of a single time scale does not necessarily occur, which violates a fundamental assumption of the single-oscillator theory. Even so, for the case of global coupling a modified approach for the linearized part of the dynamics has been worked out. The central prediction of that analysis is that optimal amplification occurs only near the onset of symmetry-preserving bifurcations of the in-phase state. By definition, the in-phase state corresponds to the situation where all of the oscillators have the same wave form and phase; they oscillate with perfect coherence [$\varphi_k(t) = \varphi_0(t)$ for all k]. All instabilities of the in-phase state can be classified as either symmetry preserving or symmetry breaking, depending on whether the instability preserves or destroys the coherence. (Roughly speaking, if the oscillators remain in phase after the bifurcation, this is a symmetry-preserving bifurcation; otherwise, it is a symmetry-breaking bifurcation.)

In the simulations, we focus on period doubling bifurcations of the in-phase state, which corresponds to the so-called three-photon mode: in principle this instability can be either symmetry breaking or symmetry preserving, depending on the system parameters. In the symmetry-preserving case, the amplification of a periodic input signal of frequency ω_s is predicted to follow the scaling law

$$S(\omega_s) \propto N^2 \frac{a^2}{\epsilon^2 + \delta^2} \ ,$$

where $\delta = \omega_s - \frac{1}{2}\omega$ is the detuning frequency, ϵ is the bifurcation parameter (equal to zero precisely at the bifurcation point), and N is the number of Josephson junctions in the array. This is the same scaling law as for the single-element case [3] with the important addition of the prefactor N^2, which illustrates the advantage of using arrays. (In fact, it is more appropriate to consider the power delivered to a *matched* load, i.e., a load whose impedance increases linearly with N, in which case the power delivered is proportional to N.) In contrast, near a symmetry-breaking bifurcation the array theory predicts that there is no significant amplification of the periodic perturbation. However, our simulations show that this is overly pessimistic: though the gain is not so large as near a symmetry-preserving bifurcation, it is still possible to get substantial gain near a symmetry-breaking instability. This observation is especially important since it may be impossible in a given system for the in-phase state to undergo a symmetry-preserving bifurcation. Indeed, the Josephson-junction array depicted in Fig. 1 with positive resistive load appears to be an example of this, as we now discuss.

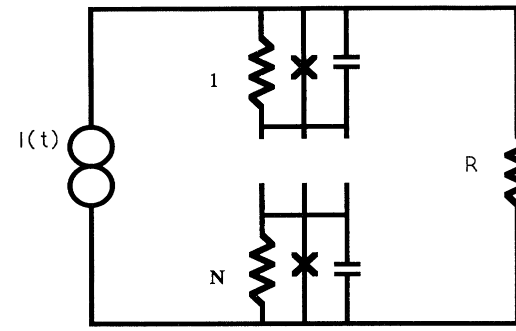

FIG. 1. Schematic of resistively shunted Josephson-junction array. Junctions are represented by Stewart-McCumber equivalent circuits (resistor, capacitor, and ideal junction in parallel).

III. FORBIDDEN BIFURCATIONS

Before turning to our numerical results, we discuss an important property of this particular dynamical system. Namely, we have found that *the in-phase attractor always undergoes a symmetry-breaking instability provided the load resistance* $R > 0$. While we have not been able to prove this statement rigorously, we can understand this behavior by the following heuristic argument.

Consider an attracting in-phase solution of the unperturbed Eq. (1) ($a = 0$) so that all $\varphi_k(t)$ are equal to the same function $\varphi_0(t)$. If we linearize Eq. (1) about this solution we get

$$\ddot{\varepsilon}_k + \gamma\dot{\varepsilon}_k + \cos[\varphi_0(t)]\varepsilon_k + \frac{1}{R}\sum_{j=1}^{N}\dot{\varepsilon}_j = 0 \ , \quad k = 1, \ldots, N \ , \quad (2)$$

where $\varepsilon_k(t) = \varphi_k(t) - \varphi_0(t)$. Now we introduce the following change of variables:

$$\eta_j = \varepsilon_j - \varepsilon_{j+1} \quad j = 1, \ldots, N-1 \ ,$$

$$H = \sum_{j=1}^{N}\varepsilon_j \ .$$

The N equations decouple with the $N - 1$ equations

$$\ddot{\eta}_k + \gamma\dot{\eta}_k + \cos[\varphi_0(t)]\eta_k = 0 \ , \quad k = 1, \ldots, N-1 \quad (3)$$

describing relative differences in voltage between adjacent oscillators and one equation

$$\ddot{H} + \left(\gamma + \frac{N}{R}\right)\dot{H} + \cos[\varphi_0(t)]H = 0 \quad (4)$$

describing the total voltage drop. Equations (3) and (4) determine the stability of the in-phase solution. Since this solution is (by assumption) an attractor, all of the $2N$ complex Floquet exponents have negative real parts. If, as a control parameter is varied, any of the exponents of Eq. (3) cross into the right half plane, the bifurcation is of the symmetry-breaking type. If exponents of Eq. (4) cross into the right half plane, the bifurcation is symmetry preserving.

The crucial observation is that these equations differ only by the presence of the parameter N/R in Eq. (4). We can think of Eqs. (3) and (4) as the same equations of motion for parametrically driven linear oscillators, with different damping parameters. Physically, the effect of having $N/R > 0$ is to increase the damping in Eq. (4), and so makes it more stable than Eq. (3). We conclude that the symmetry-preserving bifurcation is "forbidden": any instability of the in-phase attractor is necessarily symmetry breaking.

This conclusion is borne out by our numerical simulations. Moreover, this same argument suggests how we can modify the dynamical system to achieve symmetry-preserving bifurcations of the in-phase attractor. If the load resistance is negative, we expect the symmetry-breaking bifurcation to be forbidden. This is exactly what we find in our simulations. Thus, by considering both signs of R in our simulations, we are able to test the full range of theoretical predictions using Eq. (1). While our main purpose in considering the case $R < 0$ is to allow a more complete test of Ref. [24], this is not necessarily an unphysical situation. Active circuit devices known as negative impedance converters (NIC) [34, 35] can be used to mimic a negative resistance. On the other hand, existing NIC devices operate well below the very high frequencies that make Josephson-junction arrays of interest as parametric amplifiers.

IV. NUMERICAL RESULTS

We used a fourth-order Runge-Kutta scheme to integrate Eq. (1) and computed the resulting power spectra using the Parzen algorithm [36]. The parameters used in the simulations, unless otherwise noted, are as follows. For the symmetry-preserving bifurcation, $\gamma = 0.625$, $R = -5N/\gamma$, $I_{\rm dc} = 0.25$, and $I_{\rm ac} = 1.1754$. For the symmetry-breaking bifurcation, $\gamma = 0.125$, $R = N/3\gamma$, $I_{\rm dc} = 0.25$, and $I_{\rm ac} = 0.9915$. In physical terms, keeping the ratio R/N fixed corresponds to making sure the load impedance is matched to the array impedance as the array size grows [37]. We were careful to choose parameter values that kept the real part of the critical Floquet exponent ϵ close to about -0.0100 in all cases. The drive frequency was taken to be $\omega = 1.0$, so that the resonance frequency is 0.5 (in all cases considered here, the bifurcations are period doubling), and the step size in the Runge-Kutta routine is $2\pi/1024$.

Figure 2 shows the signal gain plotted versus the detuning frequency near a symmetry-preserving bifurcation for $N = 2$, 5, and 10. Here we keep ϵ fixed at -0.0100 and vary δ between -0.078125 and $+0.078\,125$. (The observed gain depends somewhat on the relative phase Δ between the signal and the pump. This effect is significant only for the smallest arrays, and so the $N = 2$ data is averaged over ten runs corresponding to different values of Δ which was enough to render the standard deviation smaller than the size of the squares.) In each case, the data were fit to the predicted scaling form $N^2/(\epsilon^2 + \delta^2)$ (the solid lines), i.e., we made a one parameter fit for the overall amplitude of the Lorentzian. As expected, these curves are symmetric about $\delta = 0$. The three curves have the same shape, but the overall gain increases as N increases. For a fixed detuning, the gains for $N = 2, 5, 10$ (measured at resonance) are in the ratio 4:25.12:107.43, which is in good agreement with the expected N^2 scaling of 4:25:100.

Figure 3 shows the results in the case where the array is tuned close to a symmetry-breaking bifurcation. The prediction of the theory is that no significant amplification takes place [24]; however, the figure shows curves similar to Fig. 2, albeit with lower overall amplification factors. We can understand this behavior by noting that while the symmetry-breaking exponent is $\epsilon_{\rm SB} = -0.0100$, the symmetry-preserving exponent is $\epsilon_{\rm SP} = -0.106$. In other words, for these parameter values the system is in some sense simultaneously close to both types of bifurcation, though not close enough to use $\epsilon_{\rm SP}$ to quantitatively predict the curves. This may have important practical implications, since it shows that substantial gain may

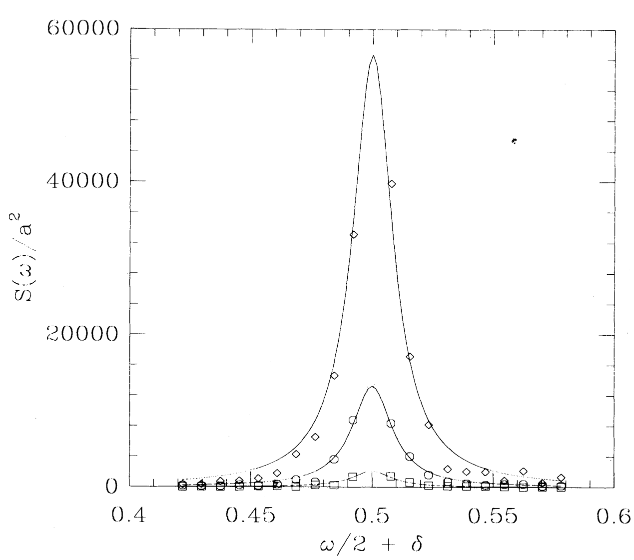

FIG. 2. Output power of array operating near symmetry-preserving bifurcation, plotted vs the detuning δ for $N = 2$, 5, and 10. □'s are $N = 2$, ○'s are $N = 5$, and ◇'s are $N = 10$. Solid lines are data fitted to the prediction. Parameters are $a = 0.001$ and $\epsilon_{\rm SP} = -0.0100$.

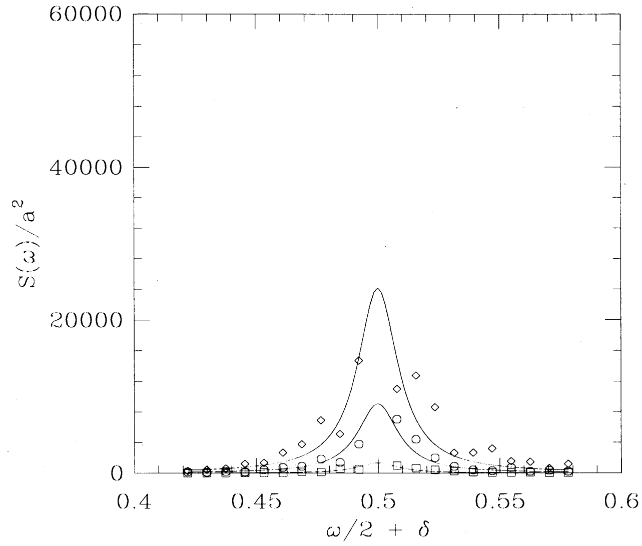

FIG. 3. Output power of array operating near symmetry-breaking bifurcation, plotted vs the detuning δ for $N = 2$, 5, and 10. □'s are $N = 2$, ∘'s are $N = 5$, and ⋄'s are $N = 10$. Parameters are $a = 0.001$ and $\epsilon_{SB} = -0.0100$.

still be possible even when the symmetry-preserving bifurcation is forbidden.

To investigate this effect further, in Fig. 4 we compare the symmetry-preserving bifurcation considered earlier with a "degenerate" case with $\epsilon_{SP} = -0.0100$ and $\epsilon_{SB} = -0.0062$. We plot the normalized response as a function of the signal frequency for array size $N = 10$. This situation is not treated in the theory paper. Naively, one might hope that the results near symmetry-preserving and symmetry-breaking bifurcations are simply additive for sufficiently small signals, but this is not quite what we see. Rather, the gain for the degenerate case is down by a factor of about 1.6; however, considerable amplification is still achieved. We conclude that while the optimal amplification associated with a symmetry-preserving bifurcation may not be achievable in the resistively shunted Josephson-junction array, large amplification is still possible by using a large enough shunt resistance so that $\epsilon_{SP} \approx \epsilon_{SB}$. Of course, larger shunt resistances also diminish the power delivered to the load.

Finally, we turn to the case of nonidentical junctions. To our knowledge, there is no existing analytic theory of amplification by such "imperfect" arrays, despite its obvious practical importance. This problem involves a number of subtleties (see below) which deserve careful study; however, our immediate concern is limited to the question of whether amplification by identical arrays is qualitatively different from arrays with just a small amount of disorder. Some degradation is expected, of course, but since disorder breaks the underlying symmetry of the dynamical system, one might worry that the disordered array is a fundamentally different problem. To investigate this, we introduced a spread in the junction parameters characterized by a single parameter, as follows. For a given N and mean value of γ, to break the symmetry by, for instance, 10%, we assign the γ's to N evenly spaced increments between 0.9γ and 1.1γ. In Fig. 5 we show gain curves for three different cases: 0%, 1%, and 10% spread in the junction parameters. In each case, the simulation was performed for a five-element array tuned near a symmetry-preserving bifurcation of the in-phase state. For the data shown, the gain at resonance is down by a factor of about 10% for a 1% spread and by 50% for a 10% spread in the parameters.

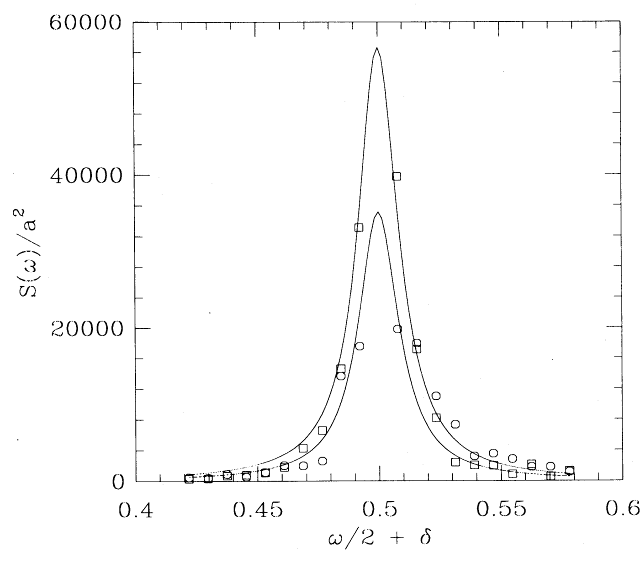

FIG. 4. Comparison of output power for symmetry-preserving case and degenerate (both symmetry-preserving and symmetry breaking) case. Parameters are $N = 10$, $a = 0.001$, $\epsilon_{SP} = -0.0100$ for the symmetry-preserving case, represented by □'s; and $\epsilon_{SP} = -0.0100$, $\epsilon_{SB} = -0.00621$ for the degenerate case, represented by ∘'s.

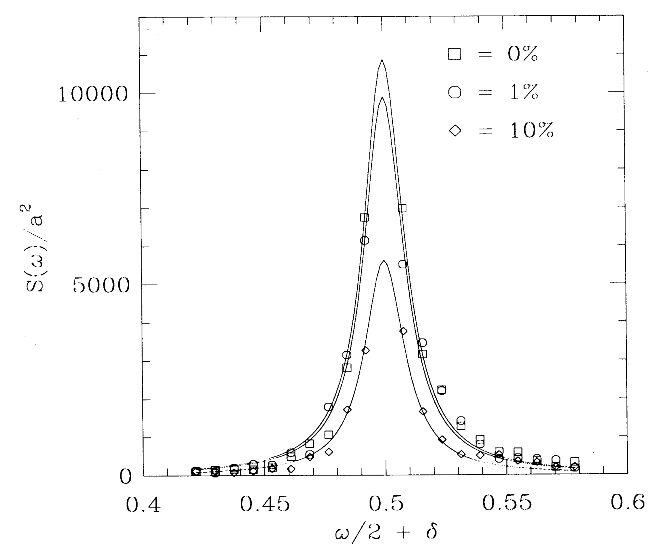

FIG. 5. Comparison of output power for three different parameter spreads, □'s for 0%, ∘'s for 1%, and ⋄'s for 10%. The parameter I_{ac} is tuned so that in all cases, the arrays have $\epsilon_{SP} = -0.0100$. The array has five junctions and is being operated near a symmetry-preserving bifurcation.

We reiterate that the introduction of disorder involves some subtle theoretical issues, which we now discuss. First, with nonidentical elements there is no true "in-phase state" where all the elements have precisely the same wave form. Similarly, there is no true symmetry in the problem to be preserved or broken by dynamical instabilities. For small amounts of disorder, however, we found that we could still identify the nearest thing to the ideal in-phase state, and that the Floquet exponents of this state divided into two groups in close correspondence with the perfect array. In other words, there is a solution branch which is the continuation of the exact in-phase state as one "turns on" the disorder. We retain the terminology used in the identical-element problem for convenience; however, such direct correspondences become problematic for high levels of disorder. Another complication is that the introduction of disorder affects the Floquet exponents: for our particular choice of disorder, increasing the spread in parameters moves the critical Floquet exponent toward the imaginary axis, i.e., moves the system closer to the bifurcation point. For example, with the fixed value $I_{\rm ac} = 1.1754$, corresponding to $\epsilon_{\rm SP} = -0.0100$ at a 0% spread in the parameters, the array will actually reach a bifurcation at about an 8% spread. Therefore holding $I_{\rm ac}$ fixed and increasing the disorder can result in an increase in gain. A more meaningful comparison is to vary the parameter spread while keeping $\epsilon_{\rm SP}$ fixed, which we accomplished by simultaneously varying $I_{\rm ac}$. For the data shown in Fig. 5, we used: $I_{\rm ac} = 1.1754$ for 0%, 1.1749 for 1%, and 1.1064 for 10%, all corresponding to a fixed value of $\epsilon_{\rm SP} = -0.0100$.

We conclude that developing a good theory of the effects of disorder on amplifier arrays is an interesting but subtle problem; a truly systematic numerical study must go well beyond the simple simulations presented in Fig. 5. Nevertheless, these data give qualitative evidence that the scaling behavior for the identical array is a meaningful limit, and this limit (which is analytically tractable) may be a good starting point for a perturbation theory that incorporates quenched disorder.

V. CONCLUSION

Our simulations tested the analytic linearized theory for small signal amplification in globally coupled oscillator arrays [24], using the dynamical equations for a one-dimensional series Josephson-junction array in parallel with a simple load. In all the cases we considered, the unperturbed array has an attracting in-phase state. Our results are in good agreement with the analytic predictions in a number of respects, the most important being that optimal amplification occurs when the array is tuned near a symmetry-preserving bifurcation. The gain curves follow the expected Lorentzian shape, and scale as N^2 which recommends the advantage of very large arrays, though the largest array considered in this paper was $N = 10$. Comparisons of arrays with different N were made with matched loads—that is, the ratio R/N was held fixed where R is the load resistance—so that the power delivered to the load scales linearly with N in the optimal case.

To our surprise, we found that the Josephson-junction array with resistive load is unable to undergo the desired symmetry-preserving bifurcation. Even so, we discovered that significant amplification could be achieved near a symmetry-breaking bifurcation, though the overall gain level was never as high as that for the corresponding symmetry-preserving bifurcation. Our theoretical understanding of the origin of the "forbidden bifurcation" for this array configuration led us to consider a negative impedance load, which indeed displayed the symmetry-preserving bifurcation.

Finally, we tested the robustness of the amplification by considering an array with nonidentical elements. We found that a spread of a few percent in the McCumber parameter gradually degraded the array performance, and did not abruptly extinguish the gain, even though dynamics has precise symmetry only in the case of zero disorder. Despite its obvious practical importance, little is known about the dynamics of nonidentical elements embedded in an oscillator array. Further work on this topic is under way.

ACKNOWLEDGMENTS

This research was supported by a grant from the Office of Naval Research. We also thank the Georgia Tech High Performance Computing Group for the use of the RS/6000 computer cluster, and the Pittsburgh Super Computing Center for the use of their Cray YMP.

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